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A REMARK TO KUIPER'S THEOREM

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1. INTRODUCTION

A REMARKABLE theorem of Kuiper asserts that the unitary group over a separable infinite dimensional Hilbert space is always contractable [3]. This fact has many important consequences in functional analysis. For instance, every Hilbert bundle is always trivial so that its associated elementary C^*-algebra is isomorphic to the trivial tensor product of the algebra of continuous functions on the base space vanishing at the infinity with the algebra of compact operators. This idea was used by P. Green [2] to prove a splitting theorem about the transformation C^*-algebra $C_\infty(M)\times_\alpha G$. A question naturally arises as to whether one has a similar splitting theorem for the smooth crossed product algebra. Namely, as a topological algebra, is the smooth crossed product algebra $C^\infty(M)\times_\alpha G$ isomorphic to the tensor product of the algebra of smooth functions over the quotient space M/G with the algebra of smooth kernels on G. To answer this question, it seems that one needs an analogy of Kuiper's theorem in the smooth context. However, at this moment, we are unable to show this. Instead, in this article, as a first step in this direction, we prove a weaker theorem which basically asserts that for certain types of groups the "regular representation" is homotopic to the identity in the smooth "unitary group" over $C^\infty(G)$. More precisely, let A be the topological algebra of smooth functions on a compact Lie group G with the inductive topology, and $\mathscr{L}(A, A)$ the vector space of all continuous linear mappings of A into A with the bounded convergence topology [4], i.e., the σ-topology with σ equal to the family of all bounded subsets of A. In other words, the topology of $\mathscr{L}(A, A)$ is generated by the family of semi-norms:

$$u \to p_{s,k}(u) = sup_{x\in S} p_k[u(x)], \quad u \in \mathscr{L}(A, A),$$

where $\{p_k, k \in N\}$ is the family of semi-norms generating the topology of A and S is the family of all bounded subsets of A.

Let $U^\infty(A)$ denote the subset of $\mathscr{L}(A, A)$ which consists of all elements $u \in \mathscr{L}(A, A)$ such that $u^{-1} \in \mathscr{L}(A, A)$ and u can be extended to a unitary operator on $L^2(G)$. We define a topology in $U^\infty(A)$ by letting a net $u_\tau \to u_0$ in $U^\infty(A)$ if and only if $u_\tau \to u_0$ and $u_\tau^{-1} \to u_0^{-1}$ in $\mathscr{L}(A, A)$. Then $U^\infty(A)$ becomes a topological group under this topology.

By ρ, we denote the continuous map from G to $U^\infty(A)$ given by $(\rho(x)f)(y) = f(x^{-1}y)$, for any $f \in A$, and $x, y \in G$. When f considered as an element in $L^2(G)$, ρ is exactly the usual regular representation of G. For this reason, ρ is still called the "regular representation" in our context. Our main theorem in this article is the following:

THEOREM 1.1. *If the Lie group G belongs to one of the following types*:

(1) *Simply connected semi-simple compact Lie groups*;

(2) *classical groups*;

(3) *tori*,

then ρ *is homotopic to id with the image of e keeping fixed at id, where e is the unit of G.*

2. PROOF OF THE MAIN THEOREM

Let G be a semi-simple compact Lie group, and $\mathscr{G}$ its Lie algebra. It follows from the root decomposition that $\mathscr{G} = h \oplus \Sigma_{\alpha\in\Delta}\mathscr{G}_\alpha$, where h is the Cartan subalgebra of $\mathscr{G}$.

Let D be the dominant, analytically integral linear functionals on $h^{\mathbb{C}}$ [1] [6]. By fixing an orthonormal basis in $(h_{\mathbb{R}})'(h_{\mathbb{R}} = ih)$, D can be identified with a lattice on $\mathbb{R}^n$, where $n = \dim_{\mathbb{R}} h$. By ρ, we denote the projection from $\mathbb{R}^n$ onto its last $(n-1)$ components and set $D_1 = p(D)$. Then D_1 is a lattice on $\mathbb{R}^{n-1}$. For any $\theta \in D_1$, let $S_\theta = \{t \in \mathbb{R} \mid te_1 + \theta \in D\}$, where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$.

First, we need to prove the following:

PROPOSITION 2.1. *Assume under a certain orthonormal basis of* $(h_{\mathbb{R}})'$ *that for any* $\theta \in D_1$,

(1) S_θ *is an increasing infinite sequence* $\{m_i\}_{i=1}^{\infty}$ $(m_1 < m_2 < \cdots)$; *and*

(2) *there exists a constant* Δ *independent of* θ *such that* $0 < m_{i+1} - m_i < \Delta$.

Then the "regular representation" ρ *is homotopic to id with the image of e keeping fixed at id.*

Before the proof of this proposition, we need the following lemma.

LEMMA 2.1. *Under the same assumption as in Proposition 2.1, we have*

(1) *there exists a constant C such that* $d(\lambda) \le C|\lambda|^m$ *for any* $\lambda \in D$, *where m is the number of positive roots, and* $d(\lambda)$ *is the degree of the representation*;

(2) *for any* $\theta \in D_1$, *let* $\lambda_i = m_i e_1 + \theta$ *and* $d_i = d(\lambda_i)$. *Then* $i \le d_i < d_{i+1}$;

(3) *there exists a constant* C_0, $(C_0 \ge 1)$, *independent of the choice of* θ, *such that*

$$\frac{\| m_{i+1} e_1 + \theta \|}{\| m_i e_1 + \theta \|} \le C_0,$$

provided that $m_i e_1 + \theta \ne 0$.

Proof.

(1) is a direct consequence of the Weyl dimension formula.

(2) It follows from the Weyl dimension formula that

$$d(\lambda) = \frac{\prod_{\alpha\in\Delta^+}(\lambda + \delta, \alpha)}{\prod_{\alpha\in\Delta^+}(\delta, \alpha)}.$$

For any $\theta \in D_1$, let $\lambda_t = te_1 + \theta$, $\forall t \in \mathbb{R}$. Then, for any $\alpha \in \Delta^+$,

$$(\lambda_t + \delta, \alpha) = t(e_1, \alpha) + (\theta, \alpha) + (\delta, \alpha).$$

Since α is a positive root,

$$\frac{d}{dt}(\lambda_t + \delta, \alpha) = (e_1, \alpha) \ge 0.$$

In fact, there exists at least one $\alpha \in \Delta^+$, such that $(e_1, \alpha) > 0$. Otherwise, we would have

$\Delta^+ \subset span(e_2, \ldots, e_n)$, which contradicts to the fact that $\dim_{\mathbb{R}} h = n$. Hence, $d(\lambda_t)$ is a strictly increasing function of $t \in \mathbb{R}$; therefore we have $d_i < d_{i+1}$. Moreover, since d_i are integers and $d_1 \geq 1$, we have $d_i \geq i$.

(3) Write $\Delta_i = m_{i+1} - m_i$. Then

$$\begin{aligned}\left|\frac{m_{i+1}^2 + |\theta|^2}{m_i^2 + |\theta|^2} - 1\right| &= \left|\frac{(m_i + \Delta_i)^2 + |\theta|^2}{m_i^2 + |\theta|^2} - 1\right| \\ &\leq \left|\frac{2m_i^2 + 2\Delta_i^2 + |\theta|^2 - (m_i^2 + |\theta|^2)}{m_i^2 + |\theta|^2}\right| \\ &= \frac{m_i^2 + 2\Delta_i^2}{m_i^2 + |\theta|^2} \\ &= \frac{m_i^2}{m_i^2 + |\theta|^2} + \frac{2\Delta_i^2}{m_i^2 + |\theta|^2} \\ &\leq 1 + 2\Delta^2/K,\end{aligned}$$

where $K = \min_{\lambda \in D - \{0\}} |\lambda|^2$. Therefore,

$$\frac{\| m_{i+1} e_1 + \theta \|}{\| m_i e_1 + \theta \|} \leq \sqrt{2 + 2\Delta^2/K} = C_0. \qquad \text{Q.E.D.}$$

Let ρ' be the usual representation of G on the Hilbert space $L^2(G)$. According to the Peter–Weyl theorem, ρ' can be expressed as follows. The Hilbert space $H = L^2(G)$ has a decompositon $H = \oplus_{\lambda \in D} \oplus_{i=1}^{d(\lambda)} H_i^\lambda$, with $\dim H_i^\lambda = d(\lambda)$, and under this decomposition ρ': $G \to U(H)$ is given by

$$\rho' \colon x \to \bigoplus_{\lambda \in D} \underbrace{(\pi_\lambda(x) \oplus \ldots \pi_\lambda(x))}_{d(\lambda)} \in \bigoplus_{\lambda \in D} U(H_1^\lambda \oplus \ldots \oplus H_{d(\lambda)}^\lambda).$$

Accordingly, the regular representation ρ in our context has a similar expression simply through the identification of $A = C^\infty(G)$ with $D(\hat{G})$ (the space of Schwarz class function on G) [5], i.e.,

$$D(\hat{G}) = \left\{ a = \bigoplus_{\lambda \in D} a^\lambda | a^\lambda \in M_{d(\lambda)}(\mathbb{C}) s.t. \forall k \in N, \quad |\lambda|^k \| a^\lambda \| \text{ is bounded for all } \lambda \in D \right\}.$$

The corresponding topology in $D(\hat{G})$ is generated by the family of seminorms p_k given by:

$$p_k(a) = sup_{\lambda \in D}(|\lambda|^k \| a^\lambda \|), \text{ where } a = \bigoplus_{\lambda \in D} a^\lambda.$$

The following lemma is an immediate consequence of Lemma 2.1.

LEMMA 2.2. *Let* $a = \oplus_{\lambda \in D} a^\lambda$ *with* a^λ *equal to* $\oplus_{i=1}^{d(\lambda)} a_i^\lambda \in M_{d(\lambda)}(\mathbb{C})$ *and* $a_i^\lambda \in H_i^\lambda$. *Suppose that* a_{ij}^λ $(1 \leq i, j \leq d(\lambda))$ *are the components of* a_i^λ *in* H_i^λ. *Then,* $a \in D(\hat{G})$ *if and only if for each* $k \in N$, *there exists a constant* C_k *such that*

(1) $|\lambda|^k \| a_i^\lambda \| \leq C_k$ *for all* $\lambda \in D$ *and* $1 \leq i \leq d(\lambda)$, *or*
(2) $|\lambda|^k |a_{ij}^\lambda| \leq C_k$, *for all* $\lambda \in D$ *and* $1 \leq i, j \leq d(\lambda)$.

With the above preliminaries, we may prove Proposition 2.1 below.

Proof of Proposition 2.1. We divide our proof into three steps.

Step 1. ρ is homotopic to ρ_1 in $\oplus_{\lambda\in D} U(H_1^\lambda \oplus \ldots \oplus H_{d(\lambda)}^\lambda)$. Here ρ_1 is given by

$$\rho_1(x) = \bigoplus_{\lambda\in D} \underbrace{((\pi_\lambda(x))^{d(\lambda)} \oplus 1 \oplus \ldots \oplus 1)}_{d(\lambda)}$$

$$\in \bigoplus_{\lambda\in D} U(H_1^\lambda \oplus \cdots \oplus H_{d(\lambda)}^\lambda), \text{ for any } x \in G,$$

where 1 denotes the identity map. In fact, the homotopy can be obtained by applying the following formula repeatedly:

$$W(t) = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos(\pi/2)t & \sin(\pi/2)t \\ -\sin(\pi/2)t & \cos(\pi/2)t \end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos(\pi/2)t & -\sin(\pi/2)t \\ \sin(\pi/2)t & \cos(\pi/2)t \end{pmatrix}, \quad (1)$$

with

$$W(0) = \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} \quad (2)$$

and

$$W(1) = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}. \quad (3)$$

Let $W_\lambda(t, x)$ be the homotopy between $\underbrace{(\pi_\lambda(x) \oplus \ldots \oplus \pi_\lambda(x))}_{d(\lambda)}$ and $\underbrace{((\pi_\lambda(x))^{d(\lambda)} \oplus 1 \oplus \ldots \oplus 1)}_{d(\lambda)}$ obtained by using Formula (1) $(d(\lambda) - 1)$ times. We claim that $W(t, x) = \oplus_{\lambda\in D} W_\lambda(t, x)$ is the desired homotopy.

In order to show this, first observe that $W(t, x)$ and $(W(t, x))^{-1} = (W(t, x))^*$ belong to $\oplus_{\lambda\in D} U(H_1^\lambda \oplus \ldots \oplus H_{d(\lambda)}^\lambda)|_{D(\hat{G})}$, which is a subset of $\mathscr{L}(A, A)$ according to Lemma 2.2. As for the continuity of $W(t, x)$ we need the following observation.

For any unitary $n \times n$ matrices u and v, if $W(t)$ is the homotopy given by Formula (1), then

$$\| W(t) - W(s) \| \leq \sqrt{2}\pi |s - t|, \quad \forall s, t.$$

This estimate is proven as follows: set

$$W_1(t) = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos(\pi/2)t & \sin(\pi/2)t \\ -\sin(\pi/2)t & \cos(\pi/2)t \end{pmatrix}$$

and

$$W_2(t) = \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos(\pi/2)t & -\sin(\pi/2)t \\ \sin(\pi/2)t & \cos(\pi/2)t \end{pmatrix}.$$

Then

$$\begin{aligned} \| W_1(t) - W_1(s) \| &= \left\| \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos(\pi/2)t - \cos(\pi/2)s & \sin(\pi/2)t - \sin(\pi/2)s \\ -\sin(\pi/2)t + \sin(\pi/2)s & \cos(\pi/2)t - \cos(\pi/2)s \end{pmatrix} \right\| \\ &\leq \left\| \begin{pmatrix} \cos(\pi/2)t - \cos(\pi/2)s & \sin(\pi/2)t - \sin(\pi/2)s \\ -\sin(\pi/2)t + \sin(\pi/2)s & \cos(\pi/2)t - \cos(\pi/2)s \end{pmatrix} \right\| \\ &\leq \frac{\sqrt{2}\pi}{2} |t - s|. \end{aligned}$$

Similarly, $\| W_2(t) - W_2(s) \| \le \frac{\sqrt{2}\pi}{2} |t - s|$. Therefore,

$$\begin{aligned}
\| W(t) - W(s) \| &= \| W_1(t) W_2(t) - W_1(s) W_2(s) \| \\
&= \| W_1(t)(W_2(t) - W_2(s)) \| + \|(W_1(t) - W_1(s)) W_2(s) \| \\
&\le \| W_2(t) - W_2(s) \| + \| W_1(t) - W_1(s) \| \\
&\le \sqrt{2}\pi |t - s|.
\end{aligned}$$

As an immediate consequence of this observation, we have

$$\| W_\lambda(t, x) - W_\lambda(s, x) \| \le \sqrt{2}\pi d(\lambda) |t - s|, \quad \forall x \in G.$$

For any bounded subset $S \subseteq A$, suppose that $a = \oplus_{\lambda \in D} a^\lambda$ is an arbitrary element in S. Then for any $k \in N$,

$$\begin{aligned}
|\lambda|^k \| (W_\lambda(t, x) - W_\lambda(s, x)) a^\lambda \| &\le |\lambda|^k \| W_\lambda(t, x) - W_\lambda(s, x) \| \, \| a^\lambda \| \\
&\le \sqrt{2}\pi |\lambda|^k d(\lambda) \| a^\lambda \| \, |t - s| \\
\text{(by Lemma 2.1(1))} &\le C\sqrt{2}\pi |\lambda|^{k+m} \| a^\lambda \| \, |t - s| \\
&\le C\sqrt{2}\pi p_{k+m}(a) |t - s|.
\end{aligned}$$

Hence,

$$\begin{aligned}
p_{S,k}[W(t, x) - W(s, x)] &= sup_{a \in S} p_k[(W(t, x) - W(s, x))a] \\
&= sup_{a \in S} sup_{\lambda \in D} [|\lambda|^k \| (W_\lambda(t, x) - W_\lambda(s, x)) a^\lambda \|] \\
&\le C\sqrt{2}\pi sup_{a \in S} [p_{k+m}(a)] |t - s|.
\end{aligned}$$

Similarly,

$$\begin{aligned}
p_{S,k}[W^{-1}(t, x) - W^{-1}(s, x)] &= p_{S,k}[W^*(t, x) - W^*(s, x)] \\
&\le C\sqrt{2}\pi \, sup_{a \in S} [p_{k+m}(a)] \, |t - s|.
\end{aligned}$$

Hence, $W(t, x)$ is uniformly continuous in $U^\infty(A)$ with respect to $t \in [0, 1]$; therefore, it is a continuous homotopy.

Step 2. First, we rewrite the decomposition of $L^2(G)$ as follows:

$$\begin{aligned}
L^2(G) &= \bigoplus_{\lambda \in D} \bigoplus_{i=1}^{d(\lambda)} H_i^\lambda \\
&= \bigoplus_{\theta \in D_1} \bigoplus_{n=1}^{\infty} \bigoplus_{i=1}^{d(m_n^\theta)} H_i^{m_n^\theta},
\end{aligned}$$

where $m_n^\theta = m_n e_1 + \theta \in D$. Let

$$\begin{aligned}
H_\theta &= \bigoplus_{n=1}^{\infty} \bigoplus_{i=1}^{d(m_n^\theta)} H_i^{m_n^\theta} \\
&= \left(\bigoplus_{i=1}^{\infty} H_\theta^i \right) \oplus \tilde{H}_\theta,
\end{aligned}$$

where

$$H_\theta^1 = H_1^{m_1^\theta} \oplus H_2^{m_2^\theta} \oplus H_2^{m_3^\theta} \oplus H_2^{m_4^\theta} \oplus \ldots$$

$$H_\theta^2 = H_1^{m_2^\theta} \oplus H_3^{m_3^\theta} \oplus H_3^{m_4^\theta} \oplus H_3^{m_5^\theta} \oplus \ldots$$

$$\ldots$$

$$H_\theta^i = H_1^{m_i^\theta} \oplus H_{i+1}^{m_{i+1}^\theta} \oplus H_{i+1}^{m_{i+2}^\theta} \oplus H_{i+1}^{m_{i+3}^\theta} \oplus \ldots,$$

and

$$\tilde{H}_\theta = \bigoplus_{i=1}^{\infty} \bigoplus_{j=i+1}^{d(m_i^\theta)} H_j^{m_i^\theta}.$$

This decomposition of H_θ is always possible, since $d(m_n^\theta) = d_n \geq n$ according to Lemma 2.1. Then

$$L^2(G) = \bigoplus_{\theta \in D_1} H_\theta = \bigoplus_{\theta \in D_1} \left(\left(\bigoplus_{i=1}^{\infty} H_\theta^i \right) \oplus \tilde{H}_\theta \right). \tag{4}$$

Under the above decomposition,

$$\rho_1(x)|_{H_\theta^i} = \mathrm{diag}(\pi_{m_i^\theta}^{d(m_i^\theta)}(x), 1, 1, \ldots 1 \ldots), \text{ and}$$

$$\rho_1|_{\tilde{H}_\theta} = id.$$

On the other hand, we introduce $\rho_2 \in U^\infty(A)$ as follows. For any $x \in G$ and $\theta \in D_1$,

$$\rho_2(x)|_{H_\theta^i} = \mathrm{diag}(\pi_{m_i^\theta}^{d(m_i^\theta)}(x), \pi_{m_i^\theta}^{d(m_i^\theta)}(x^{-1}), \pi_{m_i^\theta}^{d(m_i^\theta)}(x), \pi_{m_i^\theta}^{d(m_i^\theta)}(x^{-1}), \ldots), \quad \forall i \in N, \text{ and}$$

$$\rho_2|_{\tilde{H}_\theta} = \rho_1|_{\tilde{H}_\theta}.$$

We claim that ρ_1 is homotopic to ρ_2 in $U^\infty(A)$.

Decompose H_θ^i naturally into an infinite direct sum of subspaces with each summand having dimension $d(m_i^\theta)$. In other words, we write

$$H_\theta^i = H_1^{m_i^\theta} \oplus H_{2,\theta}^i \oplus H_{3,\theta}^i \oplus \ldots$$

such that

$$\dim H_{n,\theta}^i = d(m_i^\theta), \quad \text{for } n = 2, 3, \ldots.$$

Applying Formula (1) given in the first step, we can easily achieve the desired homotopy in $(\bigoplus_{i=1}^\infty \mathrm{U}_\theta^i) \oplus 1$ by taking $u = \pi_{m_i^\theta}^{d(m_i^\theta)}(x)$ and $v = \pi_{m_i^\theta}^{d(m_i^\theta)}(x^{-1})$, where

$$\mathrm{U}_\theta^i = 1 \oplus U(H_{2,\theta}^i \oplus H_{3,\theta}^i) \oplus U(H_{4,\theta}^i \oplus H_{5,\theta}^i) \oplus \cdots.$$

It remains to show that $(\bigoplus_{i=1}^\infty \mathrm{U}_\theta^i) \oplus 1 \subset U^\infty(A)$ and the above homotopy is continuous. Let $u = (\bigoplus_{i=1}^\infty u_\theta^i) \oplus 1$ be an arbitrary element in $(\bigoplus_{i=1}^\infty \mathrm{U}_\theta^i) \oplus 1$ and $b = u(a)$ for any $a \in D(\hat{G})$. Under Decomposition (4), we can write

$$a = \bigoplus_{\theta \in D_1} \left(\bigoplus_{i=1}^{\infty} \left(\sum_j a_{1,j}^{m_i^\theta} e_{1,j}^{m_i^\theta} \oplus \sum_j a_{i+1,j}^{m_{i+1}^\theta} e_{i+1,j}^{m_{i+1}^\theta} \oplus \sum_j a_{i+1,j}^{m_{i+2}^\theta} e_{i+1,j}^{m_{i+2}^\theta} \oplus \ldots \right) \oplus \tilde{a}_\theta \right)$$

$$b = \bigoplus_{\theta \in D_1} \left(\bigoplus_{i=1}^{\infty} \left(\sum_j b_{1,j}^{m_i^\theta} e_{1,j}^{m_i^\theta} \oplus \sum_j b_{i+1,j}^{m_{i+1}^\theta} e_{i+1,j}^{m_{i+1}^\theta} \oplus \sum_j b_{i+1,j}^{m_{i+2}^\theta} e_{i+1,j}^{m_{i+2}^\theta} \oplus \ldots \right) \oplus \tilde{b}_\theta \right).$$

Then all b's can be written as linear combinations of a's as given in the following:

$$b_{i+1,j}^{m_{i+v}^\theta} = \sum_{t=1}^{2d(m_i^\theta)} (*)\, a_{i+1,*}^{m_{i+v}^\theta}, \qquad (v \geq 1), \tag{5}$$

where all the coefficients $(*)$ have absolute values not greater than one. All the other terms of b are equal to the corresponding terms of a.

Since $d(m_i^\theta)$ is increasing with respect to i according to Lemma 2.1 (2), it follows that the only possible values for s in Equation (5) are $v-3, v-2, \ldots, v+2, v+3$. Hence,

$$
\begin{aligned}
|m_{i+v}^\theta|^k |b_{i+1,j}^{m_{i+v}^\theta}| &\leq \sum_{t=1}^{2d(m_i^\theta)} |m_{i+v}^\theta|^k |a_{i+1,*}^{m_{i+s}^\theta}| \\
&= \sum_{t=1}^{2d(m_i^\theta)} \frac{|m_{i+v}^\theta|^k}{|m_{i+s}^\theta|^{k+m}} |m_{i+s}^\theta|^{k+m} |a_{i+1,*}^{m_{i+s}^\theta}| \qquad \text{(by Lemma 2.1 (3))} \\
&\leq \sum_{t=1}^{2d(m_i^\theta)} C_0^{3k} \frac{1}{|m_{i+s}^\theta|^m} C_{k+m} \qquad \text{(since } s \geq 0) \\
&\leq 2d(m_i^\theta) C_0^{3k} \frac{1}{|m_i^\theta|^m} C_{k+m} \\
&\leq 2CC_0^{3k} C_{k+m}.
\end{aligned}
$$

Therefore, $b \in D(\hat{G})$. In fact, the above argument shows that $u \in \mathscr{L}(A, A)$. Similarly, we have $u^{-1} = u^* \in \mathscr{L}(A, A)$. Hence, $u \in U^\infty(A)$. Finally, it can be proved similarly as in Step 1 that this homotopy from ρ_1 to ρ_2 is continuous.

Step 3. Similarly as in Step 2, we can prove that ρ_2 is continuously homotopic to 1 in $(\bigoplus_{i=1}^\infty \hat{U}_\theta^i) \oplus 1$, where

$$\hat{U}_\theta^i = U(H_1^{m_i^\theta} \oplus H_{2,\theta}^i) \oplus U(H_{3,\theta}^i \oplus H_{4,\theta}^i) \oplus \ldots$$

and

$$\left(\bigoplus_{i=1}^\infty \hat{U}_\theta^i \right) \oplus 1 \subset U^\infty(A). \qquad \text{Q.E.D.}$$

Below, we should show that simply connected semi-simple compact Lie groups and classical groups automatically satisfy the two conditions in Proposition 2.1.

PROPOSITION 2.2. *Conditions* (1) *and* (2) *in Proposition* 2.1 *are satisfied by any simply connected semi-simple compact Lie groups.*

Proof. Let G be a simply connected semi-simple compact Lie group and $\mathscr{G}$ its Lie algebra. Suppose that $\{e'_1, \ldots, e'_n\}$ is a simple roots system of $\mathscr{G}$. From this roots system, by using Schmidt process, we can obtain an orthonormal basis $\{e_1, \ldots, e_n\}$ of $(h_{\mathbb{R}})'$.

First of all, we claim that under this basis $\{e_1, \ldots, e_n\}$, if $t_0 \in S_\theta$ then $t_0 + k\|e'_1\| \in S_\theta$, for any $k \in N$.

It is known that analytically integral functionals coincide with algebraically integral functionals for simply connected semi-simple Lie groups. Therefore, it follows that

$$t_0 \in S_\theta \text{ iff } \frac{2(t_0 e_1 + \theta, e'_i)}{\|e'_i\|^2} \in \mathbb{Z},$$

and

$$\frac{2(t_0 e_1 + \theta, \alpha)}{\|\alpha\|^2} \geq 0, \quad \forall \alpha \in \Delta^+.$$

Suppose that $t_0 \in S_\theta$; then for any $k \in N$ and $\alpha \in \Delta^+$,

$$\frac{2((t_0 + k\|e'_1\|)e_1 + \theta, \alpha)}{\|\alpha\|^2} = \frac{2(t_0 e_1 + \theta, \alpha)}{\|\alpha\|^2} + 2k\|e'_1\| \frac{(e_1, \alpha)}{\|\alpha\|^2}$$

$$\geq 0 \quad \text{(since } \alpha \text{ is a positive root)},$$

and

$$\frac{2((t_0 + k\|e'_1\|)e_1 + \theta, e'_i)}{\|e'_i\|^2} = \frac{2(t_0 e_1 + \theta, e'_i)}{\|e'_i\|^2} + k\frac{2(e'_1, e'_i)}{\|e'_i\|^2}$$

$$= \frac{2(t_0 e_1 + \theta, e'_i)}{\|e'_i\|^2} + kA_{i1} \in \mathbb{Z},$$

where $(A_{i,j})$ is the Cartan matrix of $\mathscr{G}$. Therefore, $t_0 + k\|e'_1\| \in S_\theta$. Now, Conditions (1) and (2) are immediate consequences of this claim. In fact, for Condition (2), we have $0 < m_{i+1} - m_i \leq \|e'_1\|$. Q.E.D.

The following result is well-known (see [1][6]).

PROPOSITION 2.3. *All the classical groups:*

$$SU(n),\ U(n),\ Sp(n),\ SO(2n) \text{ and } SO(2n+1)$$

satisfy Conditions (1) *and* (2) *in Proposition* 2.1.

Finally, it remains to investigate the case of n-tori.

PROPOSITION 2.4. *If G is a n-tori T^n, then the "regular representation" ρ is homotopic to id.*

Proof. By Fourier transformation, $L^2(T^n)$ can be decomposed as

$$L^2(T^n) = \underbrace{l^2 \otimes l^2 \otimes \ldots \otimes l^2}_{n},$$

with $\{e_{i_1} \otimes \ldots \otimes e_{i_n} | i_1, \ldots, i_n \in \mathbb{Z}\}$ being an orthonormal basis, where $e_k(t) = e^{-ikt}$. Under this basis, the "regular representation" ρ: $T^n \to U^\infty(A)$ can be written as

$$\rho(z)(e_{i_1} \otimes \ldots \otimes e_{i_n}) = z_1^{i_1} \ldots z_n^{i_n} (e_{i_1} \otimes \ldots \otimes e_{i_n}), \text{ for any } z = (z_1, \ldots, z_n) \in T^n.$$

Therefore,

$$\rho(T^n) \subset 1 \oplus \bigoplus_{i_1, \ldots, i_n \text{ not all } 0} SU(\mathbb{C} \cdot e_{i_1} \otimes \ldots \otimes e_{i_n} \oplus \mathbb{C} \cdot e_{-i_1} \otimes \ldots \otimes e_{-i_n}).$$

In fact,

$$\rho(z) = 1 \oplus \bigoplus_{i_1, \ldots, i_n \text{ not all } 0} \operatorname{diag}(z_1^{i_1} \ldots z_n^{i_n}, z_1^{-i_1} \ldots z_n^{-i_n}).$$

By applying Formula (1), we can show similarly as above that ρ is continuously homotopic to *id*, and in fact this homotopy can be realized in

$$1 \oplus \bigoplus_{i_1, \ldots, i_n \text{ not all } 0} SU(\mathbb{C} \cdot e_{i_1} \otimes \ldots \otimes e_{i_n} \oplus \mathbb{C} \cdot e_{-i_1} \otimes \cdots \otimes e_{-i_n}) \subset U^\infty(A).$$

Q.E.D.

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